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Quantum Stationary State of Class A Bianchi Universe

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Padmanabhan derived a differential equation for the stationary state for the class A Bianchi model and obtained some approximate solutions. Here we reduce the differential equation to a standard, well-known, solvable linear differential equation and indicate some exact explicit particular solutions.

1. INTRODUCTION

Padmanabhan (1984) obtained the stationary state equation for the class A Bianchi model as given by

$$-\frac{\hbar^2}{4q^2}\left(\frac{\partial^2}{\partial u_1^2}+\frac{\partial^2}{\partial u_2^2}\right)\psi+\frac{\hbar^2}{4}\frac{\partial^2\psi}{\partial q^2}-cq^{2/3}\exp(-c_2u_1)\psi=E\psi \tag{1.1}$$

where

$$\begin{aligned}
u_1&=\left(\frac{3}{8}\right)^{1/2}\left[\frac{3}{2}(1-3h)\right]^{1/2}[\beta_1+(-3h)^{1/2}\beta_2]\\
u_2&=\left(\frac{3}{8}\right)^{1/2}\left[\frac{3}{2}(1-3h)\right]^{1/2}[(-3h)^{1/2}\beta_1-\beta_2]\\
q&=\left(\frac{8}{3}\right)^{1/2}e^{3\lambda/2}\\
c_1&=\left(\frac{3}{8}\right)^{1/2}\frac{2A^2(3h-1)}{h}\\
c_2&=\left(\frac{8}{3}\right)^{1/2}\left[\frac{8}{3(1-3h)}\right]^{1/2}
\end{aligned} \tag{1.2}$$

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Here h is the Binachi parameter, A is a constant, β_1, β_2, and λ are functions of time, and $\psi(\lambda, \beta_1, \beta_2, t)$ is the wave function. One can separate out the u_2 dependence by assuming

$$\psi(u_1, u_2, q) = \exp(iku_2)\phi(u_1, q)$$

Then $\phi(u_1, q)$ satisfies the equation

$$\left(-\frac{\partial^2\phi}{\partial u_1^2} + q^2\frac{\partial^2\phi}{\partial q^2}\right) - \frac{4c_1}{\hbar^2} q^{8/3} \exp(-c_2 u_1)\phi = \left(\frac{4E}{\hbar^2} q^2 - k^2\right)\phi \qquad (1.3)$$

Padmanabhan (1984) obtained the solutions of equation (1.3) near $q = 0$. In this paper, we shall reduce equation (1.3) to a standard, well-known, solvable linear differential equation and indicate some exact explicit particular solutions. We hope this will facilitate the study of the physical aspects of the problem.

2. REDUCTION TO A STANDARD, WELL-KNOWN, SOLVABLE LINEAR PARTIAL DIFFERENTIAL EQUATION

Let us put $r = \text{in } q$, so that

$$\phi_q = (1/q)\phi_r, \qquad q^2\phi_{qq} = \phi_{rr} - \phi_r \qquad (2.1)$$

and equation (1.3) becomes

$$(-\phi_{u_1u_1} + \phi_{rr} - \phi_r) - \frac{4c_1}{\hbar^2} \exp\left[-c_2\left(u_1 - \frac{8r}{3c_2}\right)\right]\phi = \left[\frac{4E}{\hbar^2}\exp(2r) - k^2\right]\phi \qquad (2.2)$$

Putting

$$r = w, \qquad u_1 - 8r/3c_2 = v$$

so that

$$\phi_{u_1} = \phi_v, \qquad \phi_{u_1u_1} = \phi_{vv}$$
$$\phi_r = \phi_w - (8/3c_2)\phi_v \qquad (2.3)$$

and

$$\phi_{rr} = \phi_{ww} - (16/3c_2)\phi_{wv} + (64/9c_2^2)\phi_{vv}$$

we find that equation (2.2) reduces to

$$\phi_{ww} + \left(\frac{64}{9c_2^2} - 1\right)\phi_{vv} - \frac{16}{3c_2}\phi_{wv} - \phi_w + \frac{8}{3c_2}\phi_v + \left(k^2 - \frac{4c_1}{\hbar^2}e^{-c_2v} - \frac{4E}{\hbar^2}e^{2w}\right)\phi = 0 \qquad (2.4)$$

If we now set

$$e^{2w} = x, \qquad e^{-c_2 v} = y$$

we have

$$\begin{aligned} \phi_w &= 2x\phi_x, \qquad \phi_{ww} = 4x^2\phi_{xx} + 4x\phi_x \\ \phi_v &= c_2 y\phi_y, \qquad \phi_{vv} = c_2^2 y^2 \phi_{yy} + c_2^2 y\phi_y \\ \phi_{wv} &= -2c_2 xy\phi_{xy} \end{aligned} \tag{2.5}$$

Substituting (2.5) in equation (2.4), we obtain

$$4x^2\phi_{xx} + \left(\frac{64}{9} - c_2^2\right) y^2 \phi_{uu} + \frac{32}{3} xy\phi_{xy} - 2x\phi_x$$

$$+ \left(\frac{40}{9} - c_2^2\right) y\phi_y + \left(k^2 - \frac{4E}{\hbar^2}x - \frac{4c_1}{\hbar^2}y\right)\phi = 0 \tag{2.6}$$

We now introduce the following notations, which are common in the theory of partial differential equations:

$$\begin{aligned} p &= \frac{\partial \phi}{\partial x}, \qquad q = \frac{\partial \phi}{\partial y}, \qquad r = \frac{\partial^2 \phi}{\partial x^2} \\ s &= \frac{\partial^2 \phi}{\partial x\, \partial y}, \qquad t = \frac{\partial^2 \phi}{\partial y^2} \end{aligned} \tag{2.7}$$

Written in terms of *r*, *s*, and *t*, equation (2.6) reads

$$Ar + 2Bs + ct = F \tag{2.8}$$

where

$$\begin{aligned} A &= 4x^2, \qquad B = (16/3)xy, \qquad C = (64/9 - c_2^2)y \\ F &= F(x, y, \phi), \partial\phi/\partial x\, \partial\phi/\partial y) \end{aligned} \tag{2.9}$$

Equation (2.8) is a standard linear partial differential equation and it is possible to solve it exactly (Forsyth, 1956; Sommerfeld, 1949). We now mention only the important points for the solvability of equation (2.8) for certain boundary conditions and the behavior of ϕ.

Let us assume that both ϕ and the derivative $\partial\phi/\partial n$ of ϕ in the direction of the normal are prescribed along a curve Γ in the xy plane. In the theory of partial differential equations, the following relations are valid in general, and therefore hold on Γ:

$$dp = r\, dx + s\, dy \tag{2.10a}$$

$$dq = s\, dx + t\, dy \tag{2.10b}$$

Now, since p and q are known on Γ, equations (2.8), (2.10a) and (2.10b) constitute three linear equations for the determination of r, s, and t on the curve Γ. The determinant of the system is

$$\Delta = \begin{vmatrix} A & 2B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = A\,dy^2 - 2B\,dx\,dy + C\,dx^2$$

Only when this determinant Δ is different from zero can r, s, and t be calculated from (2.8), (2.10a), and (2.10b). However, in general, two directions dy, dx exist for every point (x, y) for which this is not the case. Therefore two (real or complex conjugate) families of curves exist on which $\Delta = 0$ and which according to Monge are called characteristics (Forsyth, 1956; Sommerfeld, 1949). Along each of these characteristics it is in general impossible to solve r, s, and t in terms of ϕ, p, and q. We shall therefore demand a necessary condition for solvability of equation (2.8) that Γ shall be nowhere tangent to a characteristic. The opposite case, in which Γ coincides with any of the characteristics, is connected with D'Alembert's solution.

Now we turn to the determination of the characteristics of equations (2.10a) and (2.10b). The equation of the characteristics is

$$\begin{aligned} &A\,dy^2 - 2B\,dx\,dy + C\,dx^2 \\ &\quad = \{A\,dy - [B + (B^2 - AC)^{1/2}]\,dx\} \\ &\qquad \times \{A\,dy - [B - (B^2 - AC)^{1/2}]\,dx\} \\ &\quad = 0 \end{aligned} \tag{2.11}$$

The appropriate discriminant for our problem is

$$B^2 - AC = \left(\frac{16}{3}xy\right)^2 - 4x^2\left(\frac{64}{9} - c_2^2\right)y^2 = 4x^2y^2c_2^2 \tag{2.12}$$

where $c_2^2 = 64/9(1-3h)$. Now, from the value of the discriminant we will obtain very interesting information regarding the solution ϕ, which is our main interest.

(a) When $c_2^2 > 0$ (i.e., the value of the Bianchi parameter is less than 1/3), then the discriminant is real and equation (2.8) is of hyperbolic type in which the characteristics form two distinct families. For the hyperbolic equation there is no solution with point singularities, but it is well known that there are fundamental solutions which are singular all along the characteristics. Thus, the solution ϕ is not well behaved along the characteristics.

(b) When $c_2^2 < 0$ (i.e., the value of the Bianchi parameter is greater than 1/3), then $B^2 - AC < 0$, and thus equation (2.8) becomes of elliptic type in which the characteristics are conjugate complex. The solution ϕ is in general not well behaved.

(c) When $c_2^2=0$, then $B^2-AC=0$, and thus equation (2.8) becomes of parabolic type in which only one real family of characteristics exists. The solution ϕ is an analytic function of x and y. Now the value of the Bianchi parameter become infinity, which seems unphysical.

3. SOME EXACT EXPLICIT PARTICULAR SOLUTIONS

Let

$$\phi=\sum a_{m,n}x^m y^n \tag{3.1}$$

be a solution of equation (2.6). Then one can obtain the relation

$$a_{m,n}\left[\left(2m+\frac{8}{3}n\right)^2-6m-\frac{8}{3}n-c_2^2n^2+k^2\right]$$

$$-\frac{4E}{\hbar^2}a_{m-1,n}-\frac{4c_1}{\hbar^2}a_{m,n-1}=0 \tag{3.2}$$

Case 1. We choose the values of m and n, say $m=m_0$ and $n=n_0$, in such a way that

$$(2m_0-\tfrac{8}{3}n_0)^2-6m_0-\tfrac{8}{3}n_0-c_2^2n_0^2+k^2=0 \tag{3.5}$$

We assume that

$$\begin{aligned} a_{m,n}&=0 \qquad \text{for } m+n<m_0+n_0\\ a_{m,n}&=0 \qquad \text{for } m+n=m_0+n_0\\ &\qquad\quad \text{but } m\neq m_0,\ n\neq n_0 \end{aligned} \tag{3.4}$$

$$a_{m_0,n_0}\neq 0$$

Under the assumptions (3.3) and (3.4), one can uniquely determine the nonvanishing coefficients $a_{m,n}$ for $m+n>m_0+n_0$, i.e., the nonvanishing coefficients $a_{m,n}$ for $m+n>m_0+n_0$ as given by a_{m_0+1,n_0} and a_{m_0,n_0+1} (for $m+n=m_0+n_0+1$); a_{m_0+2,n_0}, a_{m_0+1,n_0+1}, a_{m_0,n_0+2} (for $m+n=m_0+n_0+2$); a_{m_0+3,n_0}, a_{m_0+2,n_0+1}, a_{m_0+1,n_0+2}, and a_{m_0,n_0+3} (for $m+n=m_0+n_0+3$), and so on. It is to be noted that all these nonvanishing coefficients are quite consistent. Thus, under the assumptions (3.3) and (3.4), the solution of equation (1.3) is given by

$$\begin{aligned}\phi=&\,a_{m_0,n_0}x^{m_0}y^{n_0}+a_{m_0+1,n_0}x^{m_0+1}y^{n_0}\\ &+a_{m_0,n_0+1}x^{m_0}y^{n_0+1}+a_{m_0+2,n_0}x^{m_0+2}y^{n_0}\\ &+\cdots\end{aligned} \tag{3.5}$$

where

$$x = q^2, \qquad y = \exp\left[-c_2\left(u_1 - \frac{8 \ln q}{3c_2}\right)\right] \tag{3.6}$$

In particular, if one sets $m_0 = 1$, $n_0 = 0$, then the assumptions (3.3) and (3.4) become

$$\begin{aligned} k &= \sqrt{2} \\ a_{m,n} &= 0 \qquad \text{for } m+n<1 \\ a_{m,n} &= 0 \qquad \text{for } m+n=1 \\ a_{1,0} &\neq 0 \end{aligned} \tag{3.7}$$

Under the assumption (3.7), the solution of equation (1.3) is given by

$$\phi = a_{1,0}x + a_{2,0}x^2 + a_{1,1}xy + a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{4,0}x^4 + \cdots \tag{3.8}$$

where

$$x = q^2, \qquad y = \exp\left[-c_2\left(u_1 - \frac{8 \ln q}{3c_2}\right)\right] \tag{3.9}$$

and

$$a_{1,1} = \frac{4c_1}{\hbar^2(136/9 - c_2^2)} a_{1,0}$$

$$a_{2,0} = \frac{2E}{3\hbar^2} a_{1,0}$$

$$a_{3,0} = \frac{2E^2}{15\hbar^4} a_{1,0}, \quad \text{etc.}$$

are nonvanishing coefficients and are all quite consistent.

Case 2. We first consider equation (3.2) in the form

$$(2m + \tfrac{8}{3}n)^2 - 6m - \tfrac{8}{3}n - c_2^2 n^2 + k^2$$

If we set $m = 4r$ and $n = -3r$, where r is any positive integer, the above expression becomes

$$k^2 - 16r - 9c_2^2 r^2 < 0 \qquad \text{if} \quad c_2^2 \geq 0$$

We now choose the values of m and n, say $m = m_0 = 4r_0$ and $n = n_0 = -3r_0$, r_0 being a positive integer, in such a way that $k^2 = 16r_0 + 9c_2^2 r_0$ $(c_2^2 \geq 0)$.

Now proceeding exactly as in case 1, one can also obtain a series solution of equation (1.3).

4. CONCLUSION

In summary, we have reduced the stationary state equation (1.3) for class A Bianchi model obtained by Padmanabhan (1984) to a standard, well-known, solvable linear partial differential equation given by (2.6), i.e., given by (2.8). We have mentioned only the important points for the solvability of equation (2.8) for certain boundary conditions and behavior of ϕ. We have also presented some exact explict particular solutions of equation (1.3) given by (3.5), (3.6) and (3.8), (3.9) in case 1, and we have noted another type of series solution of equation (1.3) in case 2. We hope the solutions given here will facilitate the study of the physical aspects of the problem.

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